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Geometry and $N = 2$ exceptional gauge theories

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Abstract: We find the Seiberg-Witten geometry for four dimensional $N = 2$ supersymmetric E_6 gauge theories with massless fundamental hypermultiplets, by geometrically embedding them in type II string theories compactified on Calabi-Yau threefolds. The resulting geometry completely agrees with that of recent works, which are based on the technique of $N = 1$ confining phase superpotentials. We also derive the Seiberg-Witten geometry for E_7 gauge theories with massive fundamental hypermultiplets.

Keywords: String Duality, Superstring Vacua, Gauge Symmetry, Supersymmetric Effective Theories.

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Contents

1. Introduction

In the past few years, there has been much development in our understanding of non-perturbative properties of supersymmetric gauge theories and superstring theories. On one hand, it has been found that exact results for the Coulomb branch of $N = 2$ gauge theories in four dimensions can be obtained by considering auxiliary Riemann surfaces [1]. On the other hand, it has been also recognized that D-branes play the role of solitonic objects, and realize enhanced gauge symmetries especially in type II string theories compactified on singular manifolds [2]. It would be tempting to put these together, namely to embed $N = 2$ gauge theories in string theories and conjecture that the Riemann surfaces on field theory side originate from compactifying manifolds on string theory side.

Calabi-Yau threefold compactification of type II string theories indeed provides a systematic way of finding exact solutions of four dimensional $N = 2$ supersymmetric gauge theories [3, 4, 5]. However, a low energy effective theory of type II string theories contains not only gauge theory degrees of freedom, but also a gravity multiplet. In Calabi-Yau compactification approach, the gauge theory fields propagate near the singularities of the Calabi-Yau, as contrasted to the gravity multiplet which propagates on the entire Calabi-Yau space. We can therefore decouple gravity effects and consider pure gauge theories by focusing on the vicinity of the Calabi-Yau singularities.

If we construct a gauge theory from type IIA string theory, the Coulomb branch of the gauge theory is identified with the Kähler moduli of the compactifying Calabi-Yau. In type IIA theory, the Kähler moduli receive quantum corrections due to worldsheet instanton effects. So, one-loop and non-perturbative corrections to the Coulomb branch are not computable from type IIA perspective. This defect can be remedied by mapping the Kähler moduli of the type IIA Calabi-Yau to the complex moduli of the *mirror* Calabi-Yau compactifying type IIB theory. Since the complex moduli of the type IIB Calabi-Yau is free from quantum corrections, the exact metric on the Coulomb branch can be obtained from *classical* type IIB Calabi-Yau geometry.

In this article, we concentrate on E_6 and E_7 gauge theories with fundamental matter. For the E_6 case, the Seiberg-Witten geometry with massless fundamental hypermultiplets is derived from the mirror symmetry between type IIA and type IIB string theories. The resulting geometry coincides with that obtained in [6]. For the E_7 case, we apply various decoupling limits to the gauge theory, to obtain the geometry with massive fundamental hypermultiplets. The result is again consistent with [7], which have presented the geometry with a massless half hypermultiplet. In both cases, the Seiberg-Witten geometry has the form of an ALE fibration over 2-sphere, with the fibration data being slightly complicated due to the existence of extra matter. The appearance of ALE fibration is not specific to the present E_6 and E_7 cases. In general, the vector moduli information of an $N = 2$ gauge theory is expected to be more naturally encoded in an ALE fibration over 2-sphere, than in a one complex dimensional space [8, 9]. $N = 2$ field theories with other exceptional and some $\mathrm{SO}(N)$ gauge groups are discussed in [7, 10].

This paper is organized as follows. In section 2, we derive the Seiberg-Witten geometry for E_6 gauge theories with massless fundamental matter, by geometrically realizing them on Calabi-Yau singularities. In section 3, the Seiberg-Witten geometry is presented for E_7 gauge theories with massive fundamental matter. As a by-product, we will also find the geometry for $\mathrm{SO}(12)$ gauge theories with massive fundamentals and spinors. The last section is devoted to discussion and conclusions.

2. Geometric construction of E_6 gauge theories

In this section, we will investigate a mirror pair of type IIA and type IIB string theories compactified on Calabi-Yau threefolds. Type IIA string theory has an advantage that gauge groups and matter representations are easily identified. However, the Seiberg-Witten geometry can be directly obtained from the Calabi-Yau compactifying type IIB string theory. These are why we deal with both type IIA and type IIB string theories, not only with one of them. Throughout this section, toric geometry will play a crucial role. For details of toric geometry and its application to mirror symmetry, the reader should consult references [11, 12, 13, 14], where sections 9 and 10 of Greene's review [14] contain a basic introduction.

We consider first type IIA string theory compactified on a Calabi-Yau threefold X, which is a $K3$ fibration over $\mathbb{P}^1$ surface. Furthermore, we assume that the $K3$ fiber itself is an elliptic fibration over $\mathbb{P}^1$. Therefore the Calabi-Yau X can also be regarded as an elliptic fibration over the Hirzebruch surface $\mathbf{F}_n$, where the integer n determines how $\mathbb{P}^1$ is fibered over $\mathbb{P}^1$ in $\mathbf{F}_n$. Type IIA string theory on X in this particular class is conjectured to be dual to $E_8 \times E_8$ heterotic string theory on $K3 \times T^2$, with no Wilson line turned on T^2 [15, 16]. It is then possible to consistently take a large volume limit of T^2, and lift the duality to the six dimensional one. This heterotic dual description in six dimensions strongly suggests that the gauge symmetries and matter representations we will determine in the following are correct [17].

In order for the gauge theory resulting from this compactification of type IIA string theory to possess E_6 gauge symmetry, the $K3$ fiber has to develop an E_6 type singularity. This requirement is equivalent to that the elliptic fiber degenerates as one approaches some point on the $\mathbb{P}^1$ base of the $K3$ fiber, with the degeneration being of E_6 type. Such a singular Calabi-Yau X can be embedded in a $\mathbb{WP}^2_{1,2,3}$ bundle over $\mathbf{F}_n$. Let (x, y, z), (s, t), and (u, v) be the homogeneous coordinates on $\mathbb{WP}^2_{1,2,3}$, the $\mathbb{P}^1$ fiber of $\mathbf{F}_n$, and the $\mathbb{P}^1$ base of $\mathbf{F}_n$, respectively. The weights of them are as in table 1.

x	y	z	s	t	u	v
2	3	1	0	0	0	0
4	6	0	1	1	0	0
$2n+4$	$3n+6$	0	n	0	1	1

Table 1: Weights for $\mathbb{WP}^2_{1,2,3}$ bundle over $\mathbf{F}_n$.

Then, X is written as the hypersurface equation [17]

$$y^2 = x^3 + f(s,t;u,v)xz^4 + g(s,t;u,v)z^6\,, \tag{2.1}$$

where

$$f(s,t;u,v) = \sum_{i=3}^{I} s^i t^{8-i} f_{8+n(4-i)}(u,v)\,, \qquad g(s,t;u,v) = \sum_{j=4}^{J} s^j t^{12-j} g_{12+n(6-j)}(u,v)\,. \tag{2.2}$$

In (2.2), $f_{8+n(4-i)}$ and $g_{12+n(6-j)}$ are homogeneous polynomials of degrees specified by their subscripts. The indices I and J denote the largest values of i and j such that all the degrees appearing in (2.2) are not negative. The fact that $f(s,t;u,v)$ and $g(s,t;u,v)$ are divisible by s^3 and s^4 guarantees that the $K3$ fiber has an E_6 singularity, provided that the polynomials $f_{8+n(4-i)}$ and $g_{12+n(6-j)}$ have generic coefficients.

There are a few comments on the type IIA Calabi-Yau X given by (2.1) with (2.2). First, there may exist some singularities other than the E_6 singularity in which we are now interested, when the polynomials $f_{8+n(4-i)}$ and $g_{12+n(6-j)}$ have some special forms. If this occurs, some extra gauge theories will arise from the other singularities. We have then to decouple somehow the extra gauge theories from our E_6 gauge theory. We will argue this subtlety later in this section. Second, in addition to the E_6 vector multiplet, the Calabi-Yau X automatically incorporates $N_f \equiv n+6$ hypermultiplets in fundamental representation [17]. The hypermultiplets are localized at N_f extra singularities on the $\mathbb{P}^1$ base of $\mathbf{F}_n$, i.e. the points where the E_6 singularity of the $K3$ fiber becomes worse [17, 18, 19]. Hereafter, we will assume $-6 \le n < -2$ to ensure that the gauge theory is asymptotically free, and $N_f \ge 0$.

JHEP09(1999)020

The lower bound $n \geq -6$ is also required from the restriction that the E_6 gauge group must not be enhanced to larger gauge groups. The upper bound $n < -2$ has another geometrical meaning, as we will see later.

One can resolve the E_6 singularity of the $K3$ fiber by blowing up the ambient space, the $\mathbb{WP}^2_{1,2,3}$ bundle, at the locus $\{x = y = s = 0\}$. The smooth Calabi-Yau X obtained by the resolution admits a toric description, which is given by a polyhedron Δ whose vertices consist of the following vectors [20, 17],

$$\begin{aligned}
v_0 &= (0,0,0,0)\,,\\
v_1 &= (1,0,0,0)\,,\\
v_2 &= (0,1,0,0)\,,\\
v_3 &= (-2,-3,0,0)\,,\\
v_4 &= (0,0,1,0)\,,\\
v_5 &= (-4,-6,-1,0)\,,\\
v_6 &= (0,0,0,1)\,,\\
v_7 &= (-2n-4,-3n-6,-n,-1)\,,\\
v_8 &= (2,3,1,0)\,,\\
v_9 &= (3,5,2,0)\,,\\
v_{10} &= (4,6,3,0)\,,\\
v_{11} &= (3,4,2,0)\,,\\
v_{12} &= (2,2,1,0)\,,\\
v_{13} &= (2,3,2,0)\,.
\end{aligned} \tag{2.3}$$

As is well known in toric geometry, each vertex v_i in (2.3) is associated with a divisor D_i in X. The three vertices v_1, v_2, and v_3 describe the divisors in the $\mathbb{WP}^2_{1,2,3}$ fiber, $\{x = 0\}, \{y = 0\}$, and $\{z = 0\}$ restricted on X. Similarly, v_4, v_5, v_6, and v_7 are identified with the divisors in the $\mathbf{F}_n$ base in X, $\{s = 0\}, \{t = 0\}, \{u = 0\}$, and $\{v = 0\}$. The six vertices $v_8, \ldots, v_{13}$ represent the six exceptional divisors arising from the resolution (as we will see below, their restriction on the $K3$ fiber constitute six $\mathbb{P}^1$ surfaces, whose intersection form is nothing but the E_6 Cartan matrix). Finally, notice that the polyhedron Δ contains only one vector v_0 as its interior point. In general, for the hypersurface defined by a polyhedron to be Calabi-Yau, it is necessary that the polyhedron has a unique interior point [12]. The unique interior point corresponds to the canonical divisor of the ambient toric variety.

Our next task is to present type IIB Calabi-Yau $\tilde{X}$, which is the mirror partner of X. The Calabi-Yau $\tilde{X}$ takes the form of hypersurface equation

$$\sum_{i=0}^{13} a_i y_i = 0\,, \tag{2.4}$$

where the complex numbers a_i parametrize the complex deformation moduli of $\tilde{X}$.

The complex variables y_i are not independent of each other but obey the constraints

$$\prod_{i=0}^{13} y_i^{l_i^{(a)}} = 1\,, \tag{2.5}$$

where $l^{(a)}$ are fourteen dimensional vectors such that the following linear relations hold:

$$\sum_{i=0}^{13} l_i^{(a)}(v_i, 1) = 0\,. \tag{2.6}$$

Here, $(v_i, 1)$ denotes the five dimensional vector made by adding the fifth component 1 to the vector v_i. In the present case, the number of independent $l^{(a)}$'s is nine, hence the index a runs from 1 to 9 in (2.5) and (2.6). The explicit form of $l^{(a)}$ is given by

$$\begin{pmatrix} l_0^{(1)} & \cdots & l_0^{(9)} \\ \vdots & \ddots & \vdots \\ l_{13}^{(1)} & \cdots & l_{13}^{(9)} \end{pmatrix} = \begin{pmatrix} -6, & -12, & -6n-12, & -1, & -1, & -1, & 0, & 0, & 0 \\ 2, & 4, & 2n+4, & 1, & 0, & 0, & 0, & 1, & 0 \\ 3, & 6, & 3n+6, & 1, & 1, & 0, & 0, & 0, & 0 \\ 1, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 1, & n, & 0, & 0, & 0, & 0, & 0, & 1 \\ 0, & 1, & 0, & 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 1, & 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 1, & 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & -2, & 1, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 1, & -2, & 1, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 1, & -2, & 1, & 0, & 1 \\ 0, & 0, & 0, & 0, & 0, & 1, & -2, & 1, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0, & 1, & -2, & 0 \\ 0, & 0, & 0, & 0, & 0, & 1, & 0, & 0, & -2 \end{pmatrix}. \tag{2.7}$$

Note that the weights of the $\mathbb{WP}^2_{1,2,3}$ bundle (see table 1) appears in the 3×7 entries $l_i^{(a)}$ $(a = 1, 2, 3; i = 1, \ldots, 7)$. Moreover, in (2.7) the E_6 Cartan matrix has emerged as the 6×6 components $l_i^{(a)}$ $(a = 4, \ldots, 9; i = 8, \ldots, 13)$. In general, each linear relation $l^{(a)}$ corresponds to a curve class C^a in X. It is also well known that the component $l_i^{(a)}$ is proportional to the intersection number $C^a \cdot D_i$. In the present analysis, $l^{(4)}, \ldots, l^{(9)}$ are identified with the six blown up 2-spheres in the $K3$ fiber,[1] and $D_8, \ldots, D_{13}$ correspond to the six exceptional divisors in X. If we restrict these exceptional divisors on the $K3$ fiber of X, they become identical to the six blown up 2-spheres. Therefore, the appearance of the E_6 Cartan matrix ensures that the six exceptional $\mathbb{P}^1$'s in the $K3$ fiber lead to E_6 gauge symmetry.

[1] $l^{(1)}, l^{(2)}$, and $l^{(3)}$ are identified with three 2-cycles in X, namely the elliptic fiber, the $\mathbb{P}^1$ fiber of $\mathbf{F}_n$, and the $\mathbb{P}^1$ base of $\mathbf{F}_n$, respectively. Here, the "$\mathbb{P}^1$ base of $\mathbf{F}_n$" actually means the singular locus in X, $\{x = y = s = 0\}$.

Let us now turn to examining the type IIB hypersurface (2.4). Because of the relation $\sum_{i=0}^{13} l_i^{(a)} = 0$ following from (2.6), both of the equations (2.4) and (2.5) are invariant under the rescaling $y_i \to \lambda y_i,\ \lambda \in \mathbb{C}^*$. It is thus allowed to scale y_i's and put one of them to unity. We take here $y_{10} = 1$. Then, the constraints (2.5) can be solved by the four independent variables $x_1 \equiv y_9, x_2 \equiv y_{11}, x_3 \equiv y_{13}$, and $\zeta \equiv y_6$, that is, all other y_i's can be represented by some powers of x_1, x_2, x_3, and ζ. The type IIB mirror manifold (2.4) is thus rewritten in terms of x_1, x_2, x_3, and ζ as

$$\begin{aligned} W \equiv\ & \zeta + a_7 \frac{{x_3}^{N_f}}{\zeta} + {x_2}^3 + {x_3}^2 + 2{x_1}^2 x_3 + \\ & + a_{12}{x_2}^2 + a_8 x_1 x_2 + a_{13} x_3 + a_{11} x_2 + a_9 x_1 + a_{10} + \\ & + \frac{1}{{M_5}^{12}} {x_3}^4 + \frac{1}{{M_3}^6} {x_3}^3 + \frac{1}{M_0} x_1 x_2 x_3 = 0\,, \end{aligned} \tag{2.8}$$

where we have put $a_1 = a_4 = a_6 = 1$ and $a_2 = 2$, taking into account the rescaling degrees of freedom of x_1, x_2, x_3, and ζ. We have also defined scale parameters M_0, M_3, and M_5 by $a_0 \equiv 1/M_0$, $a_3 \equiv 1/(M_3)^6$, and $a_5 \equiv 1/(M_5)^{12}$. In order for the second line in the r.h.s. of (2.8) to become the standard form of versal deformation of the E_6 singularity, one must suitably reparametrize x_1, x_2, and x_3, and take the limit $M_0, M_3, M_5 \to \infty$. Let us introduce new variables

$$\begin{aligned} x &\equiv -(x_2 + c)\,, \\ y &\equiv -i\left(x_3 + x_1^2 + \frac{1}{2} a_{13} + \frac{1}{2M_0} x_1 x_2 \right), \\ z &\equiv x_1 + b\,, \end{aligned} \tag{2.9}$$

where b and c are some constants determined later. Substituting certain combinations of x, y, and z into x_1, x_2, and x_3 by means of (2.9), we obtain

$$\begin{aligned} W =\ & \zeta + a_7 \frac{{x_3}^{N_f}}{\zeta} - \\ & - \left(y^2 + x^3 + z^4 + w_2 x z^2 + w_5 x z + w_6 z^2 + w_8 x + w_9 z + w_{12} \right) - \\ & - \frac{1}{4{M_0}^2} x^2 z^2 + \frac{1}{M_0} x z^3 + \frac{1}{2{M_0}^2} b x^2 z + \\ & + \left(4b + \frac{1}{M_0} c \right) z^3 + \left(a_{12} - \frac{1}{4{M_0}^2} b^2 - 3c \right) x^2 + \\ & + \frac{1}{{M_5}^{12}} {x_3}^4 + \frac{1}{{M_3}^6} {x_3}^3\,, \end{aligned} \tag{2.10}$$

where it must be kept in mind that x_3 should be replaced with

$$x_3 = iy - z^2 - \frac{1}{2M_0} b x + \left(2b + \frac{1}{2M_0} c \right) z - \frac{1}{2} a_{13} - b^2 - \frac{1}{2M_0} bc + \frac{1}{2M_0} xz\,. \tag{2.11}$$

The deformation parameters w_i $(i = 2, 5, 6, 8, 9, 12)$ in (2.10) are given by

$$\begin{aligned}
w_2 &= \frac{3}{M_0}b + \frac{1}{2{M_0}^2}c, \\
w_5 &= a_8 - \frac{1}{2M_0}a_{13} - \frac{3}{M_0}b^2 - \frac{1}{{M_0}^2}bc, \\
w_6 &= a_{13} + 6b^2 + \frac{3}{M_0}bc + \frac{1}{4{M_0}^2}c^2, \\
w_8 &= a_{11} - a_8 b - 2a_{12}c + 3c^2 + \frac{1}{2M_0}a_{13}b + \frac{1}{M_0}b^3 + \frac{1}{2{M_0}^2}b^2 c, \\
w_9 &= -a_9 - 2a_{13}b + a_8 c - 4b^3 - \frac{1}{2M_0}a_{13}c - \frac{3}{M_0}b^2 c - \frac{1}{2{M_0}^2}bc^2, \\
w_{12} &= -a_{10} + \frac{1}{4}{a_{13}}^2 + a_9 b + a_{11}c + a_{13}b^2 - a_8 bc - a_{12}c^2 + c^3 + b^4 + \\
&\quad + \frac{1}{2M_0}a_{13}bc + \frac{1}{M_0}b^3 c + \frac{1}{4{M_0}^2}b^2c^2 \,. \qquad (2.12)
\end{aligned}$$

We require $b, c, a_8, \ldots, a_{13}$ to depend on M_0 so that w_i $(i = 2, 5, 6, 8, 9, 12)$ are fixed at some constants in the limit $M_0 \to \infty$. We can make all terms in the third and fourth line of (2.10) vanish in the limit $M_0 \to \infty$, by demanding that b, c, and a_{12} depend on M_0 in the following way:

$$\begin{aligned}
b &= M_0 d_2 + d_3 + \mathcal{O}\left(\frac{1}{M_0}\right), \\
c &= -4{M_0}^2 d_2 - 4M_0 d_3 + d_4 + \mathcal{O}\left(\frac{1}{M_0}\right), \\
a_{12} &= -12{M_0}^2 d_2 - 12M_0 d_3 + 3d_4 + \frac{1}{4}{d_2}^2 + \mathcal{O}\left(\frac{1}{M_0}\right), \qquad (2.13)
\end{aligned}$$

where d_2, d_3, and d_4 are arbitrary constants independent of M_0. Note that $w_2 = d_2 + \mathcal{O}(1/M_0)$, as one can see from the first equation in (2.12). Surprisingly, all the coefficients entering in the r.h.s. of (2.11) are kept finite in the limit $M_0 \to \infty$, as far as (2.12) and (2.13) are satisfied. Hence, while the second term in the r.h.s. of (2.10) gives rise to a non-vanishing contribution, the two terms proportional to $1/{M_3}^6$ and $1/{M_5}^{12}$ disappear in the limit $M_3, M_5 \to \infty$.

Eventually, we end up with the Seiberg-Witten geometry for the E_6 gauge theory of the form

$$\zeta + \frac{1}{\zeta}(\Lambda_{E_6})^{24-6N_f}[X^{\mathbf{27}}_{E_6}(x, y, z; w)]^{N_f} - W_{E_6}(x, y, z; w) = 0 \,, \qquad (2.14)$$

where we have introduced Λ_{E_6}, the dynamical scale for the theory, by making the identification $a_7 = (\Lambda_{E_6})^{24-6N_f}$. The polynomials W_{E_6} and $X^{\mathbf{27}}_{E_6}$ are given by

$$W_{E_6}(x, y, z; w) = y^2 + x^3 + z^4 + w_2 xz^2 + w_5 xz + w_6 z^2 + w_8 x + w_9 z + w_{12} \,, \quad (2.15)$$

and

$$X^{\mathbf{27}}_{E_6}(x,y,z;w) = iy - z^2 - \frac{1}{2}w_2 x - \frac{1}{2}w_6\,. \tag{2.16}$$

The expression (2.16), which controls the ALE fibration data for the theories with fundamental matter, was derived from (2.11), (2.12), and (2.13). The resultant manifold (2.14) with (2.15) and (2.16) completely agrees with that obtained in [6], which has been derived from the technique of $N = 1$ confining phase superpotentials [21, 22, 23].

Before closing this section, it is instructive to explain here why we had to take the limit $M_0, M_3, M_5 \to \infty$, to obtain the correct answer. As pointed out above, we must pay attention to the vicinity of the E_6 singularity of the type IIA $K3$ fiber, in order to appropriately ignore the effects from some other gauge theories. Since the E_6 singularity is located at the locus $\{x = y = s = 0\}$ in X, the divisors $\{z = 0\}$ and $\{t = 0\}$ which correspond to the vertices v_3 and v_5 in (2.3) are distant from the singularity. On the contrary, all other divisors in the $K3$ fiber (which are given by the vertices $v_1, v_2, v_4, v_8, \ldots, v_{13}$) intersect at least one of the exceptional divisors appearing from the singularity. Thus, we may eliminate the vertices v_3 and v_5 from (2.3), to "forget about" other gauge theories, without affecting the E_6 gauge theory localized at the E_6 singularity. In the present mirror map, the divisors v_3 and v_5 in X are mapped to the monomials $a_3 y_3 = \frac{1}{{M_3}^6} y_3$ and $a_5 y_5 = \frac{1}{{M_5}^{12}} y_5$ in the hypersurface equation (2.4) defining $\tilde{X}$. Therefore, taking the limit $M_3, M_5 \to \infty$ can be interpreted as extracting the gauge theory data encoded in the E_6 singularity, and ignoring the region far from the singularity.

In toric geometry language, removing the two vertices v_3 and v_5 amounts to decompactifying the $K3$ fiber of X. Indeed, as we depict in figure 1, the $\mathbb{WP}^2_{1,2,3}$ bundle over $\mathbb{P}^1$ which is described by $v_1, \ldots, v_5$, is decompactified into $\mathbb{C}^3$ which is described by v_1, v_2, and v_4.[2]

Since the $K3$ fiber is holomorphically embedded in the $\mathbb{WP}^2_{1,2,3}$ bundle, the $K3$ is also decompactified in this process. The divisors $\{z = 0\}$ and $\{t = 0\}$ are sent far away from the E_6 singularity. The precise meaning of the terminology "decompactification" used here is that the volumes of the fiber and the base of the elliptic $K3$ surface become relatively large compared to those of the exceptional $\mathbb{P}^1$'s. Under this condition, while the masses of the "W bosons" corresponding to D2-branes wrapped on the exceptional $\mathbb{P}^1$'s are kept finite, we can decouple the undesirable fields coming from D2-branes wrapped on the fiber and the base of the elliptic $K3$ surface.

However, infinitely enlarging the $K3$ is not enough for gravity and stringy effects to be properly decoupled. In order to completely switch off gravity and stringy effects,

[2]We examined here only the toric data for the $\mathbb{WP}^2_{1,2,3}$ bundle over $\mathbb{P}^1$, not for the entire toric variety, the $\mathbb{WP}^2_{1,2,3}$ bundle over $\mathbf{F}_n$. However, it is straightforward to check that the entire toric variety is actually decompactified into a $\mathbb{C}^3$ bundle over $\mathbb{P}^1$, if the asymptotic freedom condition $n < -2$ holds.

JHEP09(1999)020

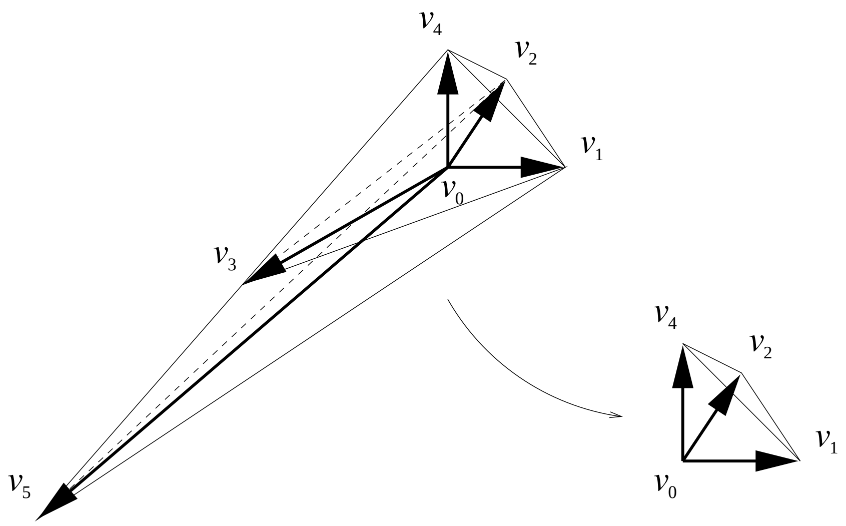

Figure 1: Decompactification of $\mathbb{WP}^2_{1,2,3}$ bundle over $\mathbb{P}^1$.

we must take the point particle limit $l_s \to 0$, where l_s is the string length. Because gauge fields live on six dimensional space (four dimensional flat space times the $\mathbb{P}^1$ base of $\mathbf{F}_n$), the four dimensional gauge coupling at the string scale, g_4, behaves as

$$\frac{1}{{g_4}^2} \sim \frac{V_b}{{l_s}^2}\,, \tag{2.17}$$

where V_b is the volume of the $\mathbb{P}^1$ base of $\mathbf{F}_n$.[3] Therefore, we must simultaneously take $V_b/l_s^2 \to \infty$, because the limit $l_s \to 0$ has to be taken so that the strong coupling scale $\Lambda_{E_6} \sim l_s^{-1}\exp(-1/{g_4}^2)$ remains finite. In general, the complexified Kähler structure associated to the curve class C^a in X is related to the complex parameters present in the type IIB mirror manifold (2.4) as

$$B^a + iV^a \sim \frac{{l_s}^2}{2\pi i}\ln\left(\prod_{i=0}^{13} a_i^{l_i^{(a)}}\right), \qquad \text{for } V^a \gg l_s^2\,, \tag{2.18}$$

where B^a denotes NS-NS 2-form background and V^a the volume of the curve C^a. We can therefore determine the behavior of V_b, when it is sufficiently large, as

$$V_b = V^3 \sim \mathrm{Im}\frac{{l_s}^2}{2\pi i}\ln\left(\frac{\Lambda_{E_6}}{M_0}\right)^{24-6N_f}, \tag{2.19}$$

[3]More precisely, the volume of the E_6 gauge symmetry locus which is represented by $l^{(3)}$ in (2.7).

where we have used (2.7) and the correspondence between 2-cycles and the vectors (2.7) given in footnote 1 (also recall that $a_1 = a_4 = a_6 = 1$ and $a_2 = 2$). Comparing (2.17) with (2.19), M_0 can be identified with the string scale l_s^{-1}. We thus conclude from (2.19) that the limit $M_0 \to \infty$ is nothing but the point particle limit $l_s \to 0$.

3. $N = 2$ E_7 gauge theories with fundamentals

In this section we consider four dimensional $N = 2$ supersymmetric E_7 gauge theories with massive fundamental hypermultiplets. Since the fundamental representation (**56**) of E_7 is pseudo-real, we can consider the case that these matter belong to half hypermultiplets. As in the previous section, we can obtain the Seiberg-Witten geometries for these theories using the toric data for E_7 [17]. The point particle limit for this case is rather trivial than the E_6 case. The result with $2N_f$ half hypermultiplets ($N_f \in \frac{1}{2}\mathbb{Z}$) are

$$\zeta + \frac{1}{\zeta}(\Lambda_{E_7})^{36-12N_f}(X_{E_7}^{\mathbf{56}})^{N_f} + W_{E_7}(x_1, x_2, x_3; w) = 0\,, \tag{3.1}$$

where $X_{E_7}^{\mathbf{56}} = {x_2}^2$ and

$$\begin{aligned} W_{E_7}(x_1, x_2, x_3; w) = {x_3}^2 + {x_2}^3 + x_2{x_1}^3 + w_2{x_2}^2x_1 + w_6{x_2}^2 + \\ + w_8x_2x_1 + w_{10}{x_1}^2 + w_{12}x_2 + w_{14}x_1 + w_{18}\,. \end{aligned} \tag{3.2}$$

The geometry for $N_f = 1/2$ case has been obtained in [7]. The deformation parameters w_i are written in terms of the Casimirs invariants constructed from an $N = 1$ adjoint chiral multiplet Φ, and the explicit relation between them is given in [24]. Note that in [24] the polynomial of the Weierstrass form

$$\tilde{W}_{E_7}(x, y, z; \tilde{w}) = y^2 + x^3 + (z^3 + \tilde{w}_8z + \tilde{w}_{12})x + \tilde{w}_2z^4 + \tilde{w}_6z^3 + \tilde{w}_{10}z^2 + \tilde{w}_{14}z + \tilde{w}_{18} \tag{3.3}$$

is used to define the deformation parameters. This is equivalent to $W_{E_7}(x_1, x_2, x_3; w)$ by the coordinate change

$$\begin{aligned} y &= x_3\,, \\ x &= x_2 + \frac{1}{3}x_1w_2 + \frac{1}{3}w_6\,, \\ z &= x_1 - \frac{1}{9}w_2^2\,, \end{aligned} \tag{3.4}$$

and the redefinition of the Casimirs

$$
\begin{aligned}
\tilde{w}_2 &= -\frac{1}{3}w_2, \\
\tilde{w}_6 &= -\frac{1}{3}\,w_6 - \frac{2}{27}\,{w_2}^3, \\
\tilde{w}_8 &= w_8 - \frac{1}{27}\,{w_2}^4 - \frac{2}{3}\,w_2\,w_6, \\
\tilde{w}_{10} &= w_{10} + \frac{1}{9}\,{w_2}^2 w_6 - \frac{1}{3}\,w_2\,w_8, \\
\tilde{w}_{12} &= w_{12} - \frac{2}{729}\,{w_2}^6 + \frac{1}{9}\,{w_2}^2 w_8 - \frac{2}{27}\,{w_2}^3 w_6 - \frac{1}{3}\,{w_6}^2, \\
\tilde{w}_{14} &= w_{14} + \frac{1}{27}\,{w_2}^4 w_6 + \frac{2}{2187}\,{w_2}^7 + \frac{2}{9}\,{w_2}^2 w_{10} - \\
&\quad - \frac{2}{27}\,{w_2}^3 w_8 + \frac{2}{9}\,w_2\,{w_6}^2 - \frac{1}{3}\,w_2\,w_{12} - \frac{1}{3}\,w_6\,w_8, \\
\tilde{w}_{18} &= w_{18} + \frac{2}{27}\,{w_6}^3 - \frac{1}{3}\,w_6\,w_{12} + \frac{5}{2187}\,{w_2}^6 w_6 + \frac{1}{19683}\,{w_2}^9 + \frac{1}{9}\,{w_2}^2 w_{14} + \\
&\quad + \frac{1}{81}\,{w_2}^4 w_{10} - \frac{1}{243}\,{w_2}^5 w_8 + \frac{2}{81}\,{w_2}^3 {w_6}^2 - \frac{1}{27}\,{w_2}^3 w_{12} - \frac{1}{27}\,w_6\,{w_2}^2 w_8\,.
\end{aligned}
\tag{3.5}
$$

If we consider the massive hypermultiplet, only the polynomial of degree twelve $X_{E_7}^{\mathbf{56}}$ should be modified in (3.1). Below we will determine $X_{E_7}^{\mathbf{56}}(x, y, z; w, m)$ for the massive case. By giving appropriate VEV's to the moduli, the geometry (3.1) should reduce to the one describing E_6 or SO(12) gauge theory as in [25].[4]

First, we consider the reduction to the E_6 gauge theory by removing the simple root α_6. For the notation for roots and weights we follow [26] and [25]. According to [25], by tuning the Higgs vector as $a^i = (M + \delta a^1, 2M + \delta a^2, 3M + \delta a^3, \frac{5}{2}M + \delta a^4, 2M + \delta a^5, \frac{3}{2}M, \frac{3}{2}M + \delta a^6)$ and taking the limit $M \to \infty$, we should obtain the E_6 singularity. Indeed by explicit calculations, we find that

$$
\tilde{W}_{E_7}(x, y, z; \tilde{w}) = (2M)^6\, W_{E_6}(x', y', z'; w'(\delta a_i)) + \mathcal{O}(M^5)\,, \tag{3.6}
$$

where

$$
\begin{aligned}
x &= 4M^2 x' \\
y &= 8M^3 y' \\
z &= -4M z' - \frac{1}{2}M^2(\tilde{w}_2 - \frac{1}{4}M^2)
\end{aligned}
\tag{3.7}
$$

and W_{E_6} is given in (2.15). The fundamental representation **56** of E_7 is decomposed into the representation of $E_6 \times \mathrm{U}(1)$ as

$$
\mathbf{56} = \mathbf{27}_{1/2} \oplus \overline{\mathbf{27}}_{-1/2} \oplus \mathbf{1}_{3/2} \oplus \mathbf{1}_{-3/2}\,, \tag{3.8}
$$

[4] SU(7) gauge theory can also be obtained in this way. But we will not consider this case because actual computation is very difficult.

JHEP09(1999)020

where the subscript denotes the U(1) charge $\alpha_6 \cdot \lambda_i$. Thus if we take $m = -\frac{1}{2}M + m'$, we should obtain the Seiberg-Witten geometry for the E_6 theory with fundamental matter in the limit $M \to \infty$. This means that

$$X_{E_7}^{\mathbf{56}}(x, y, z; w, m) = CM^6 X_{E_6}^{\mathbf{27}}(x', y', z'; w', m') + \mathcal{O}(M^5)\,, \tag{3.9}$$

where $X_{E_6}^{\mathbf{27}}$ is proposed in [6] as

$$\begin{aligned} X_{E_6}^{\mathbf{27}}(x, y, z; w, m) = \; & m^6 + 2w_2 m^4 - 8m^3 z + (w_2{}^2 - 12x)m^2 + \\ & + 4mw_5 - 4w_2 x - 8\left(z^2 - iy + \frac{w_6}{2}\right), \end{aligned} \tag{3.10}$$

and C is a constant.

Next we consider the gauge symmetry breaking which yields the SO(12) gauge theory with spinors, by giving the VEV $a^i = (2M, 3M+\delta a^5, 4M+\delta a^4, 3M+\delta a^3, 2M+\delta a^2, M+\delta a^1, 2M+\delta a^6)$ to Φ. We substitute this into $w_i(a_i)$ and look for the coordinates which eliminate the terms of the order of M^l $(9 \le l \le 18)$ in W_{E_7}. We can find such coordinates as

$$\begin{aligned} x &= \frac{1}{135}M^6 + M^4\left(\frac{1}{3}z' + \frac{1}{30}\tilde{w}_2\right) + M^2\left(\frac{1}{2}ix' + \frac{1}{10}\tilde{w}_2{}^2\right) - \frac{1}{10}\tilde{w}_6\,, \\ y &= \frac{1}{2}iM^4 y'\,, \\ z &= -\frac{1}{3}M^4 + M^2\left(\tilde{w}_2 - \frac{1}{2}z'\right) + \frac{3}{2}z'^2 - \frac{3}{2}\tilde{w}_2 z' - \tilde{w}_2{}^2\,, \end{aligned} \tag{3.11}$$

in terms of which the polynomial W_{E_7} describing the E_7 singularity is represented as

$$W_{E_7}(x, y, z; \tilde{w}) = -4M^8 W_{D_6}(x', y', z'; v) + \mathcal{O}(M^7)\,, \tag{3.12}$$

where

$$W_{D_6}(x, y, z; w) = y^2 + zx^2 + z^5 + v_2 z^4 + v_4 z^3 + v_6 z^2 + +v_8 z + v_{10} + 2ix\mathrm{Pf}\,, \tag{3.13}$$

and v_i and Pf are Casimirs of SO(12) built out of δa_i. The explicit forms of them can be read off from

$$W_{D_6}(x, y, z; w) = y^2 + zx^2 + \frac{1}{z}\left(\prod_{i=1}^{6}(z - b_i{}^2) - \prod_{i=1}^{6} b_i{}^2\right) + 2ix\prod_{i=1}^{6} b_i\,, \tag{3.14}$$

where $b_1 = \delta a_1, b_2 = \delta a_2 - \delta a_1, b_3 = \delta a_3 - \delta a_2, b_4 = \delta a_4 - \delta a_3, b_5 = \delta a_5 + \delta a_6 - \delta a_4$ and $b_6 = \delta a_6 - \delta a_5$. The fundamental representation **56** of E_7 is decomposed into the representation of SO(12) × U(1) as

$$\mathbf{56} = \mathbf{32}_0 \oplus \mathbf{10}_1 \oplus \overline{\mathbf{10}}_{-1}\,, \tag{3.15}$$

where the subscript denotes the U(1) charge $\alpha_1 \cdot \lambda_i$. The indices of spinor representation **32** and fundamental representation **10** are eight and two, respectively. Thus the terms proportional to $M^l (5 \leq l \leq 12)$ in $X^{\mathbf{56}}_{E_7}$ must be absent after taking the coordinates (x'_1, x'_2, x'_3) defined in (3.11). Note that there is no need to shift the mass to make the spinor matter survive. Although this and the condition (3.9) impose very strong restrictions on the polynomial $X^{\mathbf{56}}_{E_7}(x, y, z; w, m)$, we can find a unique solution. In the coordinates (x_1, x_2, x_3) it is

$$\begin{aligned} X^{\mathbf{56}}_{E_7}(x_1, x_2, x_3; w, m) = & \; m^{12} + 2\, w_2\, m^{10} + (6\, x_1 + {w_2}^2)\, m^8 + \\ & + (2\, x_1\, w_2 - 10\, x_2 - 4\, w_6)\, m^6 + \\ & + (-3{x_1}^2 - 6 w_2 x_2 - 4\, w_8)\, m^4 + 8\, i x_3 m^3 + \\ & + (-6\, x_2\, x_1 - 4 w_{10})\, m^2 + {x_2}^2 \,, \end{aligned} \tag{3.16}$$

and $C = 1$. Note that if we take $m = 0$, $X^{\mathbf{56}}_{E_7}$ becomes a factorized form and agrees with the massless case obtained from Calabi-Yau construction. In the semi-classical limit $\Lambda_{E_7} \to 0$, the low energy theory has singularities associated with massless squarks [6] when

$$0 = \Delta_M(m; w_i) \equiv \det_{56\times 56}(m\mathbf{1} - \Phi^{cl}) = m^{56} + 432 w_2 m^{54} + \cdots \,. \tag{3.17}$$

In fact, the hypersurface defined by the intersection of $X^{\mathbf{56}}_{E_7}(x_1, x_2, x_3; w, m) = 0$ and $W_{E_7}(x_1, x_2, x_3; w) = 0$ in $\mathbb{C}^3$ parametrized by (x_1, x_2, x_3) becomes singular when w_i's satisfy $\Delta_M = 0$. This means that in the limit $\Lambda_{E_7} \to 0$ the Seiberg-Witten geometry becomes singular when $\Delta_M = 0$. This fact can be regarded as a non-trivial check of the validity of (3.16). It is straightforward to generalize (3.1) to the theory with fundamental hypermultiplets with different masses.

We can also find the Seiberg-Witten geometry for $N = 2$ SO(12) gauge theory with spinor matter from (3.16) and (3.11). Indeed we obtain

$$X^{\mathbf{56}}_{E_7}(x_1, x_2, x_3; w; m) = M^4 X^{\mathbf{32}}_{D_6}(x', y', z'; v; m) + \mathcal{O}(M^3) \,, \tag{3.18}$$

where

$$\begin{aligned} X^{\mathbf{32}}_{D_6}(x, y, z; v; m) = & \; m^8 + (2\, z + v_2)\, m^6 + \left(3 i\, x + \frac{1}{2}\, v_2\, z + \frac{3}{8}\, {v_2}^2 - \frac{1}{2}\, v_4\right) m^4 - \\ & - 4 i\, y m^3 + \left(-2\, \mathrm{Pf} + v_6 - \frac{1}{4}\, v_4\, v_2 + \frac{1}{16}\, {v_2}^3 + 2\, v_2\, z^2 - \right. \\ & \left. - \frac{1}{8}\, z {v_2}^2 + \frac{3}{2}\, z v_4 + 3\, z^3 + 3 i\, z x + \frac{1}{2} i\, x v_2 \right) m^2 + \\ & + \frac{1}{256}\, \left(8 i\, x + 8\, z^2 + 4\, v_2\, z + 4\, v_4 - {v_2}^2\right)^2 \,. \end{aligned} \tag{3.19}$$

Taking $m = -M + m'$, which corresponds to the reduction to the SO(12) gauge theory with fundamental matter, we also obtain

$$X^{\mathbf{56}}_{E_7}(x_1, x_2, x_3; w; m) = 4 M^{10} (m'^2 - z') + \mathcal{O}(M^9) \,. \tag{3.20}$$

Therefore the Seiberg-Witten geometry for the $N=2$ SO(12) gauge theory with N_s spinors and N_f fundamentals is

$$\zeta+\frac{1}{\zeta}(\Lambda_{D_6})^{20-8N_s-2N_f}\prod_{i=1}^{N_s}X_{D_6}^{\mathbf{32}}(x,y,z;v;m_i)\prod_{j=1}^{N_f}(m'^2_j-z)+W_{D_6}(x,y,z;v)=0\,. \quad (3.21)$$

Note that in the massless case, the polynomial $X_{D_6}^{\mathbf{32}}$ is factorized and agrees with that obtained in [10]. It seems that the polynomials (3.16) and (3.19) cannot be derived from confining phase superpotential technique because they have the term of odd degrees in m.

4. Discussion and conclusions

In this article, we have studied $N=2$ supersymmetric E_6 and E_7 gauge theories with fundamental matter. For the E_7 theory, we have taken various decoupling limits such that the E_7 theory flows to the theories with smaller gauge groups, the exact solutions of which have been already known. As a result, the E_7 geometry proposed in section 3 was shown to reduce exactly to the geometries expected from earlier works. This observation serves as a non-trivial check of the validity of the original E_7 geometry. Furthermore, by breaking the group E_7 appropriately, we have also found the Seiberg-Witten geometry for SO(12) gauge theories with massive fundamentals and spinors.

We have analyzed the E_6 gauge theories, by realizing them as decoupled theories geometrically contained in type IIA string theory compactified on singular Calabi-Yau threefolds. Although the Seiberg-Witten geometry for this theory has been already constructed from field theoretical technique, it is meaningful to ascertain whether or not the same geometry can be reproduced from local structure of Calabi-Yau manifolds. Remarkably, we have found a complete agreement in the case of massless fundamental matter.

One of the disadvantages in using Calabi-Yau manifolds to determine the geometry is that the masses for matter multiplets cannot be easily incorporated. As mentioned in section 2, the matter multiplets are constrained on the extra singularities on the $\mathbb{P}^1$ base of Calabi-Yau manifolds. In order to predict how the hypermultiplet masses modify the Seiberg-Witten geometry from geometrical point of view, we presumably need to blow up the Calabi-Yau at the extra singularities. At first sight, however, it seems extremely difficult to account for the intricate mass dependence such as (3.10), (3.16), and (3.19).

Calabi-Yau construction of gauge theories has another disadvantage that we cannot realize gauge groups with arbitrarily large ranks. To overcome this difficulty, we require a more powerful framework, "geometric engineering" [3], which embeds

JHEP09(1999)020

gauge theories in non-compact Calabi-Yau threefolds instead of compact ones. Geometric engineering is presumably the most systematic and general approach to the problem of finding Seiberg-Witten geometries, even though there are several other methods which make use of branes, such as brane probe and MQCD. It is probable that understanding full geometrical features of string theories will enable us to investigate $N = 2$ supersymmetric theories with *arbitrary* gauge groups and matter representations.

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References

[1] N. Seiberg and E. Witten, *Nucl. Phys.* **B 426** (1994) 19, Erratum *ibid.* **430** (1994) 485, [`hep-th/9407087`]; *Nucl. Phys.* **B 431** (1994) 484 [`hep-th/9408099`].

[2] S. Katz, D.R. Morrison and M.R. Plesser, *Nucl. Phys.* **B 477** (1996) 105 [`hep-th/9601108`].

[3] S. Katz, A. Klemm and C. Vafa, *Nucl. Phys.* **B 497** (1997) 173 [`hep-th/9609239`].

[4] S. Katz, P. Mayr and C. Vafa, *Adv. Theor. Math. Phys.* **1** (1998) 53 [`hep-th/9706110`].

[5] For an elementary review, see P. Mayr, *Fortsch. Phys.* **47** (1999) 39 [`hep-th/9807096`].

[6] S. Terashima and S.-K. Yang, *Phys. Lett.* **B 430** (1998) 102 [`hep-th/9803014`].

[7] J.H. Brodie, *Nucl. Phys.* **B 506** (1997) 183 [`hep-th/9705068`].

[8] A. Klemm, W. Lerche, P. Mayr, C. Vafa and N. Warner, *Nucl. Phys.* **B 477** (1996) 746 [`hep-th/9604034`].

[9] W. Lerche and N.P. Warner, *Phys. Lett.* **B 423** (1998) 79 [`hep-th/9608183`].

[10] M. Aganagic and M. Gremm, *Nucl. Phys.* **B 524** (1998) 207 [`hep-th/9712011`].

[11] S. Hosono, A. Klemm, S. Theisen and S.-T. Yau, *Comm. Math. Phys.* **167** (1995) 301 [`hep-th/9308122`]; *Nucl. Phys.* **B 433** (1995) 501 [`hep-th/9406055`].

[12] V.V. Batyrev, *J. Alg. Geom.* **3** (1994) 493 [`alg-geom/9310003`].

[13] P.S. Aspinwall, B.R. Greene and D.R. Morrison, *Internat. Math. Res. Notices* (1993) 319 [`alg-geom/9309007`].

[14] B.R. Greene, *String Theory on Calabi-Yau Manifolds*, in *Fields, strings and duality, TASI 1996*, C. Esthimiou and B.R. Greene eds., World Scientific New Jersey 1997, `hep-th/9702155`.

[15] S. Kachru and C. Vafa, *Nucl. Phys.* **B 450** (1995) 69 [hep-th/9505105].

[16] P.S. Aspinwall, *K3 surfaces and string duality*, in *Fields, strings and duality, TASI 1996*, C. Esthimiou and B.R. Greene eds., World Scientific New Jersey 1997, [hep-th/9611137].

[17] M. Bershadsky, K. Intriligator, S. Kachru, D.R. Morrison, V. Sadov and C. Vafa, *Nucl. Phys.* **B 481** (1996) 215 [hep-th/9605200].

[18] S. Katz and C. Vafa, *Nucl. Phys.* **B 497** (1997) 146 [hep-th/9606086].

[19] T. Tani, to appear.

[20] P. Candelas and A. Font, *Nucl. Phys.* **B 511** (1998) 295 [hep-th/9603170].

[21] S. Elitzur, A. Forge, A. Giveon, K. Intriligator and E. Rabinovici, *Phys. Lett.* **B 379** (1996) 121 [hep-th/9603051].

[22] T. Kitao, S. Terashima and S.-K. Yang, *Phys. Lett.* **B 399** (1997) 75 [hep-th/9701009].

[23] S. Terashima and S.-K. Yang, *Nucl. Phys.* **B 519** (1998) 453 [hep-th/9706076].

[24] M. Noguchi, S. Terashima and S.-K. Yang, *Nucl. Phys.* **B 556** (1999) 115 [hep-th/9903215].

[25] S. Terashima and S.-K. Yang, *Nucl. Phys.* **B 537** (1999) 344 [hep-th/9808022].

[26] R. Slansky, *Group theory for unified model building*, *Phys. Rep.* **79** (1981) 1.

JHEP09(1999)020